International Journal of Humanities Social Science and Management (IJHSSM)
Volume 3, Issue 4, Jul.-Aug., 2023, pp: 358-362 www.ijhssm.org

A New Approach To Solve Quotient Fuzzy Stochastic Differential Equation

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Date of Submission: 19-07-2023 Date of Acceptance: 03-08-2023

ABSTRACT —The integral of the fuzzy stochastic process with respect to fuzzy Brownian motion is called the fuzzy Ito integral. Stochastic differential equation is a differential equation in which one or more term is a stochastic process and the resulting solution is also a stochastic process. This paper deals with the study of the fuzzy Ito formula for some quotient fuzzy stochastic differential equations. Using the product and quotient rule of fuzzy Ito integral, some solutions of fuzzy stochastic differential equations are discussed. Here stochastic process is a collection of fuzzy random variables and Brownian motion is a fuzzy random variable. Also, some examples are illustrated.

KEYWORD—Stochasticdifferential equation(SDE),FuzzyBrownian Motion(FBM), Fuzzy Ito integral(FII), Fuzzy stochastic process(FSP). α-cut of a fuzzy number(FN).

I. INTRODUCTION

The Differential equation is used to model the evolution of a physical system. In most cases, the coefficients of the differential equations are random. In that case, SDE will arise. So SDE is a differential equation in which one or more term is a stochastic process(SP) and the resulting solution is also a stochastic process. The application of SDE has many areas such as biology, epidemiology, mechanics, economics, finance etc. The concept of stochastic models initial-ized in 1905. After that many researchers have explained and study this model. In 1944 Ito introduced the definition of stochastic integral and the Ito formula. After that lot of researchers discussed it. In 1975 Mcshane[1] discussed a stochastic differential equations. Kloeden and Platen [2] describe the numerical solution of stochastic differential equation. Platens[3] studied the numerical methods to get the weak and strong solutions of stochastic differential equations. Saarinen et.al[4] discuss the application of stochastic differential equation for modeling the dynamic behavior of small neurons. How Brownian motion reflects on stochastic differential equation was discussed by Wang and Li[5]. Bezandry and Diagana [6] studied the concept of periodic solutions of stochastic differentials. Klebaner [7] gives a brief show of stochastic calculus and its applications. Qi [8] studies the existence and uniqueness of SDE. Wang and Liang[9] examined the use of SDE in various areas.

Because of the presence of vagueness, SDE are random as well as fuzzy. Hence, the fuzzy set theory is a useful tool for modeling this kind of equation. Zadeh [10] introduced fuzzy set theory. From that point of view, several researchers developed fuzzy logic in different areas of science, engineering and management. Also, Zadeh introduced fuzzy random variable. Then Puri, Ralescue [11] and, other researchers developed fuzzy random variable. Buckley[12] developed fuzzy random variables using fuzzy numbers as parameters in the probability distribution. Malinowski [13] study the existence of solutions to the fuzzy stochastic differential equation(FSDE). Kim[14] proves the existence and uniqueness of the solution for a FSDE using the lipschitz condition. Mitcha [15] studied the FSDE. Didier et.al [16] study the weak uniqueness of the FSDEs using FBM. Arhrrabi[17] et.al discuss the existence and uniqueness of the solution of a FSDE. Jafari[18] the non-adapted fuzzy stochastic differential using the Iterative method. Here using Buckley's [12] concept of fuzzy probabilities, some quotient SDEs are discussed. Due to the presence of randomness and fuzziness the SP, ψ_t, BM $\mathcal{B}_t$ and the coefficients of SDE $\tau(\psi_t t), \upsilon(\psi_t, t)$are taken as FSP which are denoted as $\tilde{\psi}_t, \tilde{\tau}(\tilde{\psi}_t t), \tilde{\upsilon}(\tilde{\psi}_t t)$.

The present work is organised as follows. Section-2 contains prerequisites of SDE, quotient

International Journal of Humanities Social Science and Management (IJHSSM)
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stochastic differential equations, FSDE. Some solution of quotient stochastic differential equation(QSDE) have been discussed in section-3. Finally, coclusion and concluding remarks are given in section-5 using some references.

II. BASIC PRILIMINARIES

1. Random variable[19]

It follows [19] “"A random variable is a mapping or a function from the sample space Ω onto the real line IR i.e. $\tilde{\psi}_t, t \in T$, where t is a parameter and $T \in R$".

2. Stochastic Process[19]

It follows [19] “A stochastic process is a family of random variables denoted by $\tilde{\psi}_t, t \in T$ where t is time parameter and $T \in R$".

3.Stochastic Differential equation[7]

It follows [7] “SDE is a differential equation in which one or more term is stochastic and resulting solution is also a stochastic, this equation is called SDE.

$$d\psi_t = \tau(\psi_t, t)dt + \upsilon(\psi_t, t)d\mathcal{B}(t)$$

where the coefficient $\tau(\psi_t, t)$ and $\upsilon(\psi_t, t)$ are drift and diffusion coefficients.

4. Brownian motion(BM) [7]

It follows [7]“BM $\mathcal{B}(t)$ is a continuous time SP satisfy the following property:-

(i) BM satisfies independence increment property i.e. If $t > s$ then $\mathcal{B}(t) - \mathcal{B}(s)$ is independent of past.

(ii) $\mathcal{B}$ (t) satisfies normal increment property i.e. $\mathcal{B}(t) - \mathcal{B}(s)$ has a normal distribution with mean zero and variance t-s.

(iii) $\mathcal{B}$(t) is continuous function of t.

(iv) $\mathcal{B}$(t)=$\tilde{0}$.

5. Fuzzy Brownian motion[19]

It follows [19] “A FSP $\{\widetilde{\mathcal{B}}(t), t \in [0, T], 0 < T < \infty\}$ is called a FBM on probability space (Ω, A, P) if and only if $\forall \alpha \in [0,1]$, the process

$$\widetilde{\mathcal{B}}_\alpha(t, \omega) = [\widetilde{\mathcal{B}}^L_\alpha(t, \omega), \widetilde{\mathcal{B}}^U_\alpha(t, \omega)]$$

is an interval valued BM on (Ω, A, P) and

$$\widetilde{\mathcal{B}}_\alpha(t, \omega) = \bigcup_{\alpha \in [0,1]} \widetilde{\mathcal{B}}_\alpha(t, \omega).$$

6. White Noise[7]

It follows [7] “White noise is often regarded as the informal nonexistent derivative of BM i.e. $\dot{\mathcal{B}}(t)$. K. Ito used the white noise as a random noise which is independent at different times and large fluctuation”.

7. Fuzzy Ito formula [19]

It follows [19] “Let $\tilde{\psi}_t$ be a fuzzy Ito process satisfying the FSDE

$$d\tilde{\psi}_t = \tilde{\mu}(t, \tilde{\psi}_t)\, dt \oplus (\tilde{\sigma}(t, \tilde{\psi}_t) \otimes d\tilde{B}_t),$$

and $\tilde{f}(\tilde{\psi}_t, t)$ be a twice- differentiable function, then $\tilde{f}(\tilde{\psi}_t, t)$ is a fuzzy Ito process.

Fuzzy Ito lemma is defined as

$$d(\tilde{f}(\tilde{\psi}_t, t)) = (\frac{\delta \tilde{f}}{\delta t}(\tilde{\psi}_t, t)\, d \oplus (\frac{\delta \tilde{f}}{\delta \tilde{\psi}_t} d\tilde{\psi}_t \oplus \frac{1}{2}\frac{\delta^2 \tilde{f}}{\delta \tilde{\psi}_t^2} \tilde{\sigma}^2 \tilde{\psi}^2 dt)".$$

8. Positive fuzzy number(FN)[19]

It follows [19]“A fuzzy number $\tilde{A}$ on IR is called positive FN if $\tilde{A} \succ 0$ and its the membership function $\mu_{\tilde{A}}(x)$ satisfies $\mu_{\tilde{A}}(x) = 0\ \forall x \leq 0$.

9. α-cut of an FN[15]

It follows [15] “Let $\tilde{u} \in F(IR)$ be a fuzzy number, the α-cut of $\tilde{u}$, for every $\alpha \in [0,1]$ is the set,

$$[\tilde{u}][\alpha] = \{x \in R : \tilde{u}(x) \succcurlyeq \alpha\} = [u^L[\alpha], u^U[\alpha]]\ \ (Say) \quad (1)$$

$$where\ [\tilde{u}]^L[\alpha] = \inf_{x \in IR}\{x \in [\tilde{u}][\alpha]\}$$
$$and\ [\tilde{u}]^U[\alpha] = \sup_{x \in IR}\{x \in [\tilde{u}][\alpha]\},$$

the support of $\tilde{u}$ is given by $[\tilde{u}][0] = Support(\tilde{u}) = \{x \in \text{I}R : \tilde{u}(x) \succ 0\}$.

10. Fuzzy arithmetic operations[18]

It follows [18] “α-cut of the fuzzy numbers are closed and bounded intervals. Arithmetic operation of fuzzy numbers are defined using arithmetic operations of their α-cuts. Let $\tilde{a}, \tilde{b}$ be the two fuzzy numbers whose α-cuts are $\tilde{a}[\alpha] = [a^L[\alpha], a^U[\alpha]], \tilde{b}[\alpha] = [b^L[\alpha], b^U[\alpha]]$,

Let $\oplus, \ominus, \odot$, and $\oslash$ denote addition, subtraction, multiplication, and division of fuzzy numbers.

•Fuzzy addition:-

$$(\tilde{a} \oplus \tilde{b})[\alpha] = [a^L[\alpha] + b^L[\alpha], a^U[\alpha] + b^U[\alpha]]$$

•Fuzzy subtraction:-

$$(\tilde{a} \ominus \tilde{b})[\alpha] = [a^L[\alpha] - b^L[\alpha], a^U[\alpha] - b^U[\alpha]].$$

•Fuzzy multiplication:-

$$(\tilde{a} \odot \tilde{b})[\alpha] = \tilde{a}[\alpha] \cdot \tilde{b}[\alpha]$$

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$$= [Min\{a^L[\alpha]b^L[\alpha], a^L[\alpha]b^U[\alpha], a^U[\alpha]b^L[\alpha], a^U[\alpha]b^U[\alpha]\}, Max\{a^L[\alpha]b^L[\alpha], a^L[\alpha]b^U[\alpha], a^U[\alpha]b^L[\alpha], a^U[\alpha]b^U[\alpha]\}].$$

•**Fuzzy division:-**

$$(\tilde{a} / \tilde{b})[\alpha] = \tilde{a}[\alpha]/\tilde{b}[\alpha] = [a^L[\alpha], a^U[\alpha]] \cdot [1/b^U[\alpha], 1/b^L[\alpha]],$$

$provided\ 0 \notin \tilde{b}[\alpha]$.

III. QUOTIENT FUZZY STOCHASTIC DIFFERENTIAL EQUATION

Most of the FSP are not differentiable like FBM $\tilde{\mathcal{B}}_t$ is a continuous process which is no where differentiable. So the derivatives like $\frac{d\tilde{\mathcal{B}}_t}{dt}$ does not exists in fuzzy stochastic calculus. Therefore the FSP are allowed to used the infinitesimal changes of the process in $d\tilde{\mathcal{B}}_t$.

Between time t and $t + \Delta t$, the change in fuzzy stochastic process is given by

$$\Delta \tilde{\psi}_t = \tilde{\psi}_{t+\Delta t} \ominus \tilde{\psi}_t$$

when Δt is infinitesimally small, Then the infinitesimal change of process $\tilde{\psi}_t$ can be defined as,

$$d\tilde{\psi}_t = \tilde{\psi}_{t+dt} \ominus \tilde{\psi}_t$$

Foe each $\alpha \in [0,1]$, $d\tilde{\psi}_t[\alpha] = \tilde{\psi}_{t+dt}[\alpha] - \tilde{\psi}_t[\alpha]$ is the α cut of equation(4).

Let $Let \Delta \psi_1 = \{\psi_t | \psi_t \in \tilde{\psi}_t[\alpha]\}$, $\Delta \psi_2 = \{\psi_{t+dt} | \psi_{t+dt} \in \tilde{\psi}_{t+dt}[\alpha]\}$,

$$\Delta \mathcal{B}_1 = \{\mathcal{B}_t | \mathcal{B}_t \in \tilde{\mathcal{B}}_t[\alpha]\},$$

$$\Delta \mathcal{Y} = \{\mathcal{Y}_t | \mathcal{Y}_t \in \tilde{\mathcal{Y}}_t[\alpha]\}$$

Most of the FSPes are not differentiable. Like the fuzzy Brownian motion process is a continuous process which is nowhere differentiable. Thus the derivatives like $\frac{d\tilde{\mathcal{B}}_t}{dt}$ do not make sense in stochastic calculus.

The only quantities allowed to be used are the infinitesimal changes of the process, in our case $d\tilde{\mathcal{B}}_t$.

The infinitesimal change of a process. The change in the process between instances $\tilde{t}$ and $\tilde{t} \oplus \Delta \tilde{t}$ is given by

$$\Delta \tilde{\psi}_t = \tilde{\psi}_{t+dt} \ominus \tilde{\psi}_t$$

when $\Delta \tilde{t}$ is infinitesimally small, so we obtain the infinitesimal change of process $\tilde{\psi}_t$.

$$d\tilde{\psi}_t = \tilde{\psi}_{t+dt} \ominus \tilde{\psi}_t$$

$$\tilde{\psi}_{t+dt} = \tilde{\psi}_t \oplus d\tilde{\psi}_t$$

Theorem 1 *Let $\tilde{\psi}_t$ and $\tilde{\mathcal{Y}}_t$ are two FSPs, then prove that*

(i) For a constant $\tilde{C}$ such that $d(\tilde{C}\tilde{\psi}_t) = \tilde{C}d\tilde{\psi}_t$.

(ii) $d(\tilde{\psi}_t \oplus \tilde{\mathcal{Y}}_t) = d\tilde{\psi}_t \oplus d\tilde{\mathcal{Y}}_t$.

(iii) $d(\tilde{\psi}_t \ominus \tilde{\mathcal{Y}}_t) = d\tilde{\psi}_t \ominus d\tilde{\mathcal{Y}}_t$.

(iv) $d(\tilde{\psi}_t \otimes \tilde{\mathcal{Y}}_t) = (\tilde{\psi}_t \otimes d\tilde{\mathcal{Y}}_t) \oplus (\tilde{\mathcal{Y}}_t \otimes d\tilde{\psi}_t) \oplus (d\tilde{\psi}_t \otimes d\tilde{\mathcal{Y}}_t)$.

(v)

$$d\left(\frac{\tilde{\psi}_t}{\tilde{\mathcal{Y}}_t}\right) = \frac{(\tilde{\mathcal{Y}}_t \otimes d\tilde{\psi}_t) \oplus (\tilde{\psi}_t \otimes d\tilde{\mathcal{Y}}_t) \oplus (d\tilde{\mathcal{Y}}_t \otimes d\tilde{\psi}_t)}{(\tilde{\mathcal{Y}}_t^2)} \oplus \frac{\tilde{\psi}_t}{\tilde{\mathcal{Y}}_t^3}(d\tilde{\mathcal{Y}}_t)^2$$

Proof. (i) If $\tilde{\psi}_t$ and $\tilde{\mathcal{Y}}_t$ are two stochastic processes, then $d(\tilde{\psi}_t \oplus \tilde{\mathcal{Y}}_t) = d\tilde{\psi}_t \oplus d\tilde{\mathcal{Y}}_t$.

Since $\tilde{\psi}_t$ and $\tilde{\psi}_t$ are two SPes.

(3) $\tilde{\psi}_t[\alpha] = [\tilde{\psi}_t^L[\alpha], \tilde{\psi}_t^U[\alpha]], \tilde{\mathcal{Y}}_t[\alpha] = [\tilde{\mathcal{Y}}_t^L[\alpha], \tilde{\mathcal{Y}}_t^U[\alpha]]$ are the α-cuts of fuzzy stochastic process $\tilde{\psi}_t$ and $\tilde{\mathcal{Y}}_t$ respectively.

Applying the infinitesimal change formula on$d(\tilde{\psi}_t \oplus \tilde{\mathcal{Y}}_t)$.

(4) $sod(\tilde{\psi}_t \oplus \tilde{\mathcal{Y}}_t)$

$$= (\tilde{\psi}_{t+dt} \oplus \tilde{\mathcal{Y}}_{t+dt}) - (\tilde{\psi}_t \oplus \tilde{\mathcal{Y}}_t)$$

The α-cut of $d(\tilde{\psi}_t \oplus \tilde{\mathcal{Y}}_t)$ is

$$d(\tilde{\psi}_t \oplus \tilde{\mathcal{Y}}_t)[\alpha] = \{d(\psi_t + \mathcal{Y}_t) | \Delta \psi_1, \Delta \mathcal{Y}\}$$

$$= \{(\psi_{t+dt} + \mathcal{Y}_{t+dt}) - (\psi_t + \mathcal{Y}_t) | \Delta \psi_1, \Delta \mathcal{Y}, \Delta \psi_8, \Delta \mathcal{Y}_3\}$$

$$(\Delta \mathcal{Y}_3 = \{\mathcal{Y}_{t+dt} | \mathcal{Y}_{t+dt} \in \tilde{\mathcal{Y}}_{t+dt}[\alpha]\})$$

$$= \{(\psi_{t+dt} - \psi_t) + (\mathcal{Y}_{t+dt} - \mathcal{Y}_t) | \Delta \psi_1, \Delta \mathcal{Y}, \Delta \psi_8, \Delta \mathcal{Y}_3\}$$

$$= \{d\psi_t + d\mathcal{Y}_t | \Delta \psi_1, \Delta \mathcal{Y}\}$$

$$= d\tilde{\psi}_t \oplus d\tilde{\psi}_t$$

Therefore $d(\tilde{\psi}_t \oplus \tilde{\mathcal{Y}}_t) = d\tilde{\psi}_t \oplus d\tilde{\mathcal{Y}}_t$.

(ii) If $\tilde{\psi}_t$ and $\tilde{\mathcal{Y}}_t$ are two SPes, then $d(\tilde{\psi}_t \ominus \tilde{\mathcal{Y}}_t) = d\tilde{\psi}_t \ominus d\tilde{\mathcal{Y}}_t$.

Proof. Proof is similar as proof of (ii).

(iii) If $\tilde{\psi}_t$ and $\tilde{\mathcal{Y}}_t$ are two SPes, then

$$d(\tilde{\psi}_t \otimes \tilde{\mathcal{Y}}_t) = (\tilde{\psi}_t \otimes d\tilde{\mathcal{Y}}_t) \oplus (\tilde{\mathcal{Y}}_t \otimes d\tilde{\psi}_t) \oplus (d\tilde{\psi}_t \otimes d\tilde{\mathcal{Y}}_t).$$

(5) *Proof.* Since $\tilde{\psi}_t$ and $\tilde{\psi}_t$ are two SPes.

$\tilde{\psi}_t[\alpha] = [\tilde{\psi}_t^L[\alpha], \tilde{\psi}_t^U[\alpha]], \tilde{\mathcal{Y}}_t[\alpha] = [\tilde{\mathcal{Y}}_t^L[\alpha], \tilde{\mathcal{Y}}_t^U[\alpha]]$ are the α-cuts of fuzzy SP $\tilde{\psi}_t$ and $\tilde{\mathcal{Y}}_t$ respectively.

Applying the infinitesimal change formula on $d(\tilde{\psi}_t \otimes \tilde{\mathcal{Y}}_t)$.

$$d(\tilde{\psi}_t \otimes \tilde{\mathcal{Y}}_t) = (\tilde{\psi}_{t+dt} \otimes \tilde{\mathcal{Y}}_{t+dt}) \ominus (\tilde{\psi}_t \otimes \tilde{\mathcal{Y}}_t)$$

The α-cut of $d(\tilde{\psi}_t \otimes \tilde{\mathcal{Y}}_t)$ is

$$d(\tilde{\psi}_t \otimes \tilde{\mathcal{Y}}_t)[\alpha] = \{d(\psi_t\mathcal{Y}_t) | \vartriangle \psi_1, \vartriangle \mathcal{Y}\}$$

$$= \{(\psi_{t+dt}\mathcal{Y}_{t+dt}) - (\psi_t\mathcal{Y}_t) | \vartriangle \psi_1, \vartriangle \mathcal{Y}, \vartriangle \psi_8, \vartriangle \mathcal{Y}_3\}.$$

$$= \{\psi_t(\mathcal{Y}_{t+dt} - \mathcal{Y}_t) + \mathcal{Y}_t(\psi_{t+dt} - \psi_t) + (\psi_{t+dt} - \psi_t)(\mathcal{Y}_{t+dt} - \mathcal{Y}_t) | \vartriangle \psi_1, \vartriangle \mathcal{Y}, \vartriangle \psi_8, \vartriangle \mathcal{Y}_3\}$$

$$= \{\psi_t d\mathcal{Y}_t + \mathcal{Y}_t d\psi_t + d\psi_t d\mathcal{Y}_t | \vartriangle \psi_1, \vartriangle \mathcal{Y}\}$$

Therefore $d(\tilde{\psi}_t \otimes \tilde{\mathcal{Y}}_t) = (\tilde{\psi}_t \otimes d\tilde{\mathcal{Y}}_t) \oplus (\tilde{\mathcal{Y}}_t \otimes d\tilde{\psi}_t) \oplus (d\tilde{\psi}_t \otimes d\tilde{\mathcal{Y}}_t)$.

(iv) If $\tilde{\psi}_t$ and $\tilde{\mathcal{Y}}_t$ are two stochastic processes, then $d(\frac{\tilde{\psi}_t}{\tilde{\mathcal{Y}}_t}) = \frac{(\tilde{\mathcal{Y}}_t \otimes d\tilde{\psi}_t) \oplus (\tilde{\psi}_t \otimes d\tilde{\mathcal{Y}}_t) \oplus (d\tilde{\mathcal{Y}}_t \otimes d\tilde{\psi}_t)}{(\tilde{\mathcal{Y}}_t^2)} \oplus \frac{\tilde{\psi}_t}{\tilde{\mathcal{Y}}_t^3}(d\tilde{\mathcal{Y}}_t)^2$

Proof. Since $\tilde{\psi}_t$ and $\tilde{\mathcal{Y}}_t$ are two fuzzy stochastic process.

The α-cut of $d(\frac{\tilde{\psi}_t}{\tilde{\mathcal{Y}}_t})$ is

$$d(\frac{\widetilde{\psi_t}}{\tilde{\mathcal{Y}}_t})[\alpha] = \{d(\frac{\psi_t}{\mathcal{Y}_t}) | \vartriangle \psi_1, \vartriangle \mathcal{Y}_1\}$$

$$= \{d(\psi_t.\mathcal{Y}^{-1}) | \vartriangle \psi_t, \vartriangle \mathcal{Y}_t\}$$

$$= \{\mathcal{Y}_t^{-1} d\psi_t + \psi_t d\mathcal{Y}_t^{-1} + d\psi_t d\mathcal{Y}_t^{-1} | \vartriangle \psi_t \vartriangle Y_1\},$$

$$(Using fuzzy Ito product rule)$$

$$= \{\frac{1}{\mathcal{Y}_t} d\psi_t + \psi_t[-\frac{1}{\mathcal{Y}_t^2} d\mathcal{Y}_t + \frac{1}{\mathcal{Y}^3}(d\mathcal{Y}_t)^2] + d\psi_t[-\frac{1}{\mathcal{Y}_t^2} d\mathcal{Y}_t + \frac{1}{\mathcal{Y}^3}(d\mathcal{Y}_t)^2] | \vartriangle \psi_1, \vartriangle \mathcal{Y}_1\}$$

$$[\because d(\tilde{f}(\tilde{\mathcal{Y}}_t)) = (\tilde{f}'(\tilde{\mathcal{Y}}_t) \otimes d\tilde{\mathcal{Y}}_t) + (\frac{1}{2}\tilde{f}''(\tilde{\mathcal{Y}}_t) \otimes (d\tilde{\mathcal{Y}}_t)^2)]$$

$$= \{\frac{d\psi_t}{\mathcal{Y}_t} - \frac{\psi_t}{\mathcal{Y}^2} d\mathcal{Y}_t + \frac{\psi_t}{\mathcal{Y}^3}(d\mathcal{Y}_t)^2 - \frac{d\psi_t.d\mathcal{Y}}{\mathcal{Y}^2} - \frac{d\psi_t(d\mathcal{Y}_t)^2}{\mathcal{Y}^3} | \vartriangle \psi_1, \vartriangle \mathcal{Y}_1\}$$

$$= \{\frac{d\psi_t}{\mathcal{Y}_t} - \frac{\psi_t}{\mathcal{Y}^2} d\mathcal{Y}_t - \frac{d\psi_t.d\mathcal{Y}}{\mathcal{Y}^2} + \frac{\psi_t}{\mathcal{Y}^3}(d\mathcal{Y}_t)^2 | \vartriangle \psi_1, \vartriangle \mathcal{Y}_1\}$$

$$= \{\frac{\mathcal{Y}_t d\psi_t - \psi_t d\mathcal{Y}_t - d\psi_t d\mathcal{Y}_t}{\mathcal{Y}_t^2} + \frac{\psi_t}{\mathcal{Y}_t^3}(d\mathcal{Y}_t)^2 | \vartriangle \psi_1, \vartriangle \mathcal{Y}_1\}$$

$$(\because (d\tilde{\mathcal{Y}})^2 d\tilde{\psi}_t = \tilde{0}$$

$$Hence\ d\left(\frac{\tilde{\psi}_t}{\tilde{\mathcal{Y}}_t}\right) = \frac{(\tilde{\mathcal{Y}}_t \otimes d\tilde{\psi}_t) \oplus (\tilde{\psi}_t \otimes d\tilde{\mathcal{Y}}_t) \oplus (d\tilde{\mathcal{Y}}_t \otimes d\tilde{\psi}_t)}{(\tilde{\mathcal{Y}}_t^2)} \oplus \frac{\tilde{\psi}_t}{\tilde{\mathcal{Y}}_t^3}(d\tilde{\mathcal{Y}}_t)^2.$$

Theorem 2 : *If $\tilde{\psi}_t$ is SPes and is a constant, then $d(\lambda\tilde{\psi}_t) = \lambda\, d\tilde{\psi}_t$.*

Proof. Since $\tilde{\psi}_t$ is a SP and λ is a constant.

$\tilde{\psi}_t[\alpha] = [\tilde{\psi}_t^L[\alpha], \tilde{\psi}_t^U[\alpha]]$ is the α-cut of fuzzy stochastic process $\tilde{\psi}_t$.

Applying the infinitesimal change formula on $\lambda\tilde{\psi}_t$

$$d(\lambda\tilde{\psi}_t) = \lambda\tilde{\psi}_{t+dt} \ominus \lambda\tilde{\psi}_t$$

The α-cut of $d(\lambda\tilde{\psi}_t)$ is

$$d(\lambda\tilde{\psi}_t)[\alpha] = \{d(\lambda\psi_t) | \vartriangle \psi_1\}$$

$$= \{\lambda\psi_{t+dt} - \lambda\psi_t | \vartriangle \psi_8, \vartriangle \psi_1\},$$

$$(\vartriangle \psi_8 = \{\psi_{t+dt} | \psi_{t+dt} \in \tilde{\psi}_{t+dt}[\alpha]\})$$

$$= \{\lambda(\psi_{t+dt} - \psi_t) | \vartriangle \psi_8, \vartriangle \psi_1\}$$

$$= \{\lambda d\psi_t | \vartriangle \psi_1\} = \lambda d\tilde{\psi}_t$$

Therefore $d(\lambda\tilde{\psi}_t) = \lambda d\tilde{\psi}_t$.

Lemma 1 *If $\tilde{\mathcal{B}}_t$ is a FBM then prove that*

$$d(\tilde{\mathcal{B}}_t^3) = 3(\tilde{\mathcal{B}}_t^2 \otimes d\tilde{\mathcal{B}}_t) + 3\tilde{\mathcal{B}}_t dt.$$

Proof. Using Ito product, for each $\alpha \in [0,1]$,

$$\{d(\mathcal{B}_t)^3 | \vartriangle \mathcal{B}_1\} = \{d(\mathcal{B}_t.\mathcal{B}_t^2) | \vartriangle \mathcal{B}_1\}$$

$$= \{\mathcal{B}_t d(\mathcal{B}_t^2) + \mathcal{B}_t^2 d\mathcal{B}_t + d(\mathcal{B}_t)^2 d\mathcal{B}_t | \vartriangle \mathcal{B}_1\} \quad (6)$$

$$Now\ \{d(\mathcal{B}_t)^2 | \vartriangle \mathcal{B}_1\} = \{d(\mathcal{B}_t.\mathcal{B}_t) | \vartriangle \mathcal{B}_1\}$$

$$= \{\mathcal{B}_t d\mathcal{B}_t + \mathcal{B}_t d\mathcal{B}_t + d\mathcal{B}_t d\mathcal{B}_t | \vartriangle \mathcal{B}_1\}$$

$$\{d(\mathcal{B}_t)^2 | \vartriangle \mathcal{B}_1\} = \{2\mathcal{B}_t d\mathcal{B}_t + dt | \vartriangle \mathcal{B}_t\}, \quad (\{d\mathcal{B}_t.d\mathcal{B}_t | \vartriangle \mathcal{B}_t\} = dt)$$

For each $\alpha \in [0,1]$, equation (5) is

$$= \{\mathcal{B}_t[2\mathcal{B}_t d\mathcal{B}_t + dt] + \mathcal{B}_t^2 d\mathcal{B}_t + d\mathcal{B}_t(2\mathcal{B}_t d\mathcal{B}_t + dt) | \vartriangle \mathcal{B}_1\}$$

$$= \{2\mathcal{B}_t^2 d\mathcal{B}_t + \mathcal{B}_t dt + \mathcal{B}_t^2 d\mathcal{B}_t + 2\mathcal{B}_t (d\mathcal{B}_t)^2 + d\mathcal{B}_t dt | \vartriangle \mathcal{B}_1\}$$

$$= \{3\mathcal{B}_t^2 d\mathcal{B}_t + 3\mathcal{B}_t dt | \vartriangle \mathcal{B}_1\}$$

$$(\because \{(d\mathcal{B}_t)^2 | \vartriangle \mathcal{B}_1\} = dt, and \; \{dtd\mathcal{B}_t | \vartriangle \mathcal{B}_1\} = \tilde{0})$$

Lemma 2 *If $\widetilde{\mathcal{B}}_t$ is a FBM then prove that*

$$d(t\widetilde{\mathcal{B}}_t) = \widetilde{\mathcal{B}}_t dt \oplus td\widetilde{\mathcal{B}}_t.$$

Proof. Using Ito product, for each $\alpha \in [0,1]$,

$$\{d(t\mathcal{B}_t) | \vartriangle \mathcal{B}_1 = \{\mathcal{B}_t dt + td\mathcal{B}_t + dtd\mathcal{B}_t | \quad \vartriangle \mathcal{B}_1\}$$

$$= \{\mathcal{B}_t dt + td\mathcal{B}_t | \vartriangle \mathcal{B}_1\}$$

IV. CONCLUSION

In this paper, the concept of the fuzzy probability by Buckley [11] is used. The FBM, FSP and, fuzzy Ito product rule are used to solve the Quotient FSDE.

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